Heat transport by turbulent convection

By LOUIS N. HOWARD

Department of Mathematics, Massachusetts Institute of Technology

(Received 5 April 1963)

Upper bounds for the heat flux through a horizontally infinite layer of fluid heated from below are obtained by maximizing the heat flux subject to (a) two integral constraints, the 'power integrals', derived from the equations of motion, and (b) the continuity equation. This variational problem is solved completely, for all values of the Rayleigh number R, when only the constraints (a) are imposed, and it is thus shown that the Nusselt number N for any statistically steady convective motion cannot exceed a certain value $N_1(R)$, which for large R is approximately $(3R/64)^{\frac{1}{2}}$. When (b) is included as a constraint, the variational problem is solved for large R, under the additional hypothesis that the solution has a single horizontal wave number; the associated upper bound on the Nusselt number is $(R/248)^{\frac{3}{8}}$. The mean properties of this maximizing 'flow', in particular the mean temperature and mean square temperature deviation fields, are found to resemble the mean properties of the real flow observed by Townsend; the results thus tend to support Malkus's hypothesis that turbulent convection maximizes heat flux.

1. Introduction

This paper is about the transport of heat by thermal convection in a horizontally infinite layer of fluid heated from below. As a mathematical model we use the equations of the Boussinesq approximation, for which a convenient reference is Chandrasekhar (1961), § 8; more detailed discussion is given by Mihaljan (1960) and by Spiegel & Veronis (1960). The solutions of the Boussinesq equations which are relevant to the physical problem are those which, besides satisfying appropriate boundary conditions on the faces of the fluid layer, are statistically steady in time and statistically homogeneous in the horizontal planes z = const. In particular we require the existence and constancy in time of horizontal averages of the various functions describing the flow, and of their products as well, and the vanishing of the horizontal averages of the horizontal velocity components; the horizontal average of the vertical velocity component is then zero also, as a consequence of the continuity equation. For brevity, these requirements of statistical steadiness and the existence of horizontal averages, etc., will be called the 'requirements of homogeneity'. (Homogeneity in time as well as in horizontal planes is to be understood in this expression.)

Below the critical Rayleigh number, the only solution of the Boussinesq equations satisfying the boundary conditions and the requirements of homogeneity is the purely conductive one, with no fluid motion. Above the critical

Rayleigh number this solution still exists, but is not unique; it is furthermore unstable. Experimentally, for Rayleigh numbers in excess of critical, but not too much so, a steady cellular convection occurs, and corresponding steady solutions of the Boussinesq equations have been obtained by Malkus & Veronis (1958) and Gor'kov (1958); these solutions exist for a finite range of horizontal cell sizes, but apparently only one cell size and cell pattern occurs in a given experimental situation, presumably for reasons of stability. There seems to be no strong reason to doubt the mathematical existence of such steady convective flows for all Rayleigh numbers greater than critical, but experimentally, steady convection is replaced by an unsteady motion when the Rayleigh number is increased to about ten times critical, and we must expect that all steady convective solutions of the Boussinesq equations become unstable for sufficiently high Rayleigh numbers. Malkus (1954*a*) has reported experiments which strongly suggest a sequence of successive instabilities, with associated qualitative changes in the nature of the flow, as the Rayleigh number is further increased leading ultimately to what can only be described as turbulence. It thus appears likely that even if we were able to determine all solutions of the Boussinesq equations and the boundary conditions and requirements of homogeneity, we should find, in the case of large Rayleigh number, a very considerable lack of uniqueness. This in itself would present no great problem if the average properties of all these solutions were the same, but this is certainly not the case with respect to the purely conductive and the steady convective solutions, for example, and seems equally unlikely when various unsteady solutions are also possible. In principle, the determination of which solution should be expected to be experimentally realized might be based on some sort of stability consideration (though it is difficult to formulate this idea precisely), but this seems impossible to carry through in practice. Malkus (1954*b*) suggested, as a physical hypothesis, to be tested by comparison of its consequences with experiment, that the solution actually realized is that one (or those) which leads to the largest heat flux across the layer, the temperature difference being prescribed. While this poses a problem easier than the study of the stability of all solutions, our inability even to determine the solutions, in the case of large Rayleigh number, makes it practically impossible to carry through even this approach. To produce a simpler problem, Malkus suggested a strengthened form of his hypothesis: the heat transport which actually occurs for large Rayleigh number is not only the maximum among all solutions of the Boussinesq equations, but may in fact be almost as large as the maximum obtainable when fields of velocity and temperature are admitted which no longer satisfy the Boussinesq equations, but are restricted only by the continuity equation, the boundary conditions, the requirements of homogeneity, *and the two simplest integral consequences of the Boussinesq equations*, generally called the 'power integrals'.

Now whether or not Malkus's hypothesis (in either form) is verified in reality, it is clear that the maximum heat transport consistent with the power integrals, the continuity equation, the boundary conditions, and the requirements of homogeneity, will (if it is finite) give an upper bound on the heat transport which actually occurs, and is consequently by no means without interest. If it should turn out that this upper bound does not greatly exceed the actual heat transport, we may infer that Malkus's hypothesis does have some correspondence with physical reality, and we may perhaps acquire greater confidence in the use of similar methods for the study of other problems in turbulence.

The basic problem of this paper is the determination of this upper bound on the heat transport, a problem mathematically of a most familiar type in variational calculus: maximize a functional defined by a certain integral, subject to some integral constraints. The continuity equation is really a differential equation constraint, of course, but one of a very simple form and one which leaves the class of competitors in the variational problem still very large. The problem posed by Malkus's first hypothesis, maximum heat transport subject to the full set of Boussinesq equations, the boundary conditions and the requirements of homogeneity, would (if solved) also give an upper bound on any actually occurring heat flux, in fact an upper bound which could not be worse and might well be better than the previous one. But the class of competitors is here not easy to describe precisely because we do not know all solutions of the Boussinesq equations, and the mathematical problem is therefore more difficult. On the other hand, from the physical point of view, the meaning of the hypothesis of maximum heat transport subject to the power integrals, etc., but not the Boussinesq equations, is somewhat obscure if we try to carry it any further than the simple statement that it gives an upper bound, because this upper bound may well only be achieved by 'flows' which do not satisfy the Boussinesq equations and are not really physical flows at all. In contrast to this, the original form of the hypothesis, while possibly incorrect, does at least make sense physically: the maximizing flow *could* be the real flow. However, it is possible to regard the solution of the problem of maximum heat transport subject to the power integrals, etc., as a step in a sequence of approximations to the determination of the maximum heat transport subject to the Boussinesq equations: it is not unreasonable to suppose that the successive imposition of more and more integral consequences of the Boussinesq equations as constraints on the problem of maximum heat transport will give a sequence of problems whose solutions converge in some useful sense to the solution of the problem with the full Boussinesq equations as constraints. From this point of view it appears that the problem posed above is not the natural first step. We might, for example, start by dropping the requirement of continuity, or use only one of the power integrals. It is not difficult to show that if only one of the power integrals is retained, whether or not one keeps the continuity equation, then the heat transport is unbounded; this goes too far in trying to simplify the problem. However, if both power integrals are retained but the continuity equation is dropped, a finite upper bound on the heat transport *can* be obtained, and this appears to be the natural first step in the hypothetical sequence of maximizing problems. This problem will be solved in § 3. Adding the requirement of continuity then seems to be the natural second step, and this will be taken up and partially solved in the case of large Rayleigh number in § 4. With the present approach, we obtain rigorous upper bounds on the heat transport, but we cannot expect that the maximizing fields of velocity and temperature will necessarily bear any close relation to those which really occur. On the other hand, if the upper bound on heat transport is not hopelessly too large, it would require an unreasonable amount of self-restraint not to compare the *average properties* of the maximizing fields with experimental observations. This will be done in § 5, together with a general physical discussion.

In his original theoretical paper on turbulent convection, Malkus (1954*b*) made other physical hypotheses in addition to that of maximum heat transport, notably the requirement that the mean temperature gradient should be nowhere positive, and the hypothesis that the 'smallest scale of motion' is determined by being just neutrally stable as an infinitesimal perturbation on the mean field. These hypotheses will not be used here; furthermore, the present mathematical technique is quite different from that of Malkus, and, it is hoped, is somewhat more complete. The basic physical ideas of this approach are, however, entirely those of Malkus, and my own acquaintance with this subject and interest in this problem have been largely a result of the many stimulating discussions with him which I have enjoyed over the past several years.

The two hypotheses, of maximum heat transport and the relation to the linear stability problem on the mean field, seem to be the most important physical ideas in Malkus's paper. (While the requirement of a nowhere-positive mean temperature gradient seems to play a large role in Malkus's treatment, I believe that this is more an artifact of the mathematical technique than an essential part of the physical ideas.) Townsend has suggested that it is the *second* of these that is the more fundamental, and he has shown (Townsend 1962) that some of Malkus's conclusions can be derived from it alone. If the present paper is regarded as an 'interpretation' of Malkus's theory, one must take the opposite view that it is the idea of maximum heat transport which is more fundamental; however, I do not mean to assert this, and the present paper should not be regarded as such an 'interpretation'. It is an attempt to explore in some detail the consequences of a *part* of Malkus's hypotheses, that part which seems to me to be most readily formulated as a well-defined mathematical problem and yet still retains features of physical interest.

2. Mathematical formulation

The Boussinesq equations may be written as follows:

$$\mathbf{u}_t + \mathbf{u}\,.\,\nabla\mathbf{u} + \rho^{-1}\nabla p - \alpha g T\mathbf{k} = \nu\nabla^2\mathbf{u}, \tag{1}$$

$$\nabla\,.\,\mathbf{u} = 0, \tag{2}$$

$$T^*_t + \mathbf{u}\,.\,\nabla T^* = \kappa\nabla^2 T^*, \tag{3}$$

where $\mathbf{u} = (u, v, w)$ is the velocity vector, T^* the temperature field, T the deviation of T^* from its horizontal average, ρ the mean density, α the coefficient of thermal expansion, g the acceleration of gravity, ν and κ the coefficients of kinematic viscosity and thermometric conductivity, p the deviation of the pressure from the hydrostatic pressure field corresponding to the horizontally averaged temperature, and $\mathbf{k}$ the vertical unit vector. The layer of fluid is taken to be $0 \leqslant z \leqslant d$, and the boundary conditions are that

$$T^*(0) = T_0, \quad T^*(d) = T_0 - \Delta T, \quad \mathbf{u}(0) = \mathbf{u}(d) = 0.$$

We shall use a horizontal bar to denote the horizontal average, and brackets $\langle\ \rangle$ to denote the average over the layer. Thus, for instance,

$$T^* = T + \overline{T^*}, \quad \langle wT \rangle = \frac{1}{d}\int_0^d \overline{wT}\, dz.$$

If (1) is multiplied (inner product) by $\mathbf{u}$ and averaged over the layer, one readily finds on taking account of the boundary conditions and requirements of homogeneity

$$\alpha g \langle wT \rangle = \nu \langle |\nabla \mathbf{u}|^2 \rangle, \tag{4}$$

where the notation $|\nabla \mathbf{u}|^2 \equiv |\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2$ for the viscous dissipation is used. This is the 'first power integral', so called because of its physical interpretation as the over-all balance between the rate of generation of energy by motion in the field of the buoyancy force $\alpha g T\mathbf{k}$, and the rate of dissipation of energy by viscosity.

If (3) is averaged horizontally one readily deduces

$$d\overline{wT^*}/dz = \kappa d^2 \overline{T^*}/dz^2,$$

and since $\overline{w} = 0$, $\overline{wT^*} = \overline{wT}$ so $\overline{wT} - \kappa(d\overline{T^*}/dz)$ must be constant, and equal to its average value. Thus

$$-\kappa \frac{d\overline{T^*}}{dz} = \kappa \frac{\Delta T}{d} + \langle wT \rangle - \overline{wT}. \tag{5}$$

This equation determines the mean temperature field $\overline{T^*}$ in terms of the 'deviations from the mean' w and T (the mean velocity field being zero).

The 'second power integral' can be obtained by multiplying (3) by T, averaging, and using (5). The result is

$$\kappa^{-1}[\langle wT \rangle^2 - \langle \overline{wT}^2 \rangle] + (\Delta T/d)\langle wT \rangle = \kappa \langle |\nabla T|^2 \rangle. \tag{6}$$

This relation can be given a simple physical interpretation which is most easily seen by a slightly different derivation. Within the framework of the Boussinesq approximation, the rate of generation of entropy in the fluid (as a result of heat conduction) is, per unit volume, $(\kappa\rho c/T_0^2)\,|\nabla T^*|^2$, where T_0 is the (absolute) temperature at $z = 0$ (or anywhere else in the layer, within the Boussinesq approximation), and c is the specific heat. The difference between the entropy flux out at the top and in at the bottom of the layer is

$$-\frac{\kappa\rho c}{T_0 - \Delta T}\frac{d\overline{T^*}}{dz}\bigg|_d + \frac{\kappa\rho c}{T_0}\frac{d\overline{T^*}}{dz}\bigg|_0 = -\frac{\kappa\rho c \Delta T}{T_0^2}\frac{d\overline{T^*}}{dz}\bigg|_0,$$

again within the Boussinesq approximation. Since we have a steady state these must balance

$$-\frac{\kappa\rho c}{T_0^2}\frac{d\overline{T^*}}{dz}\bigg|_0 = d\frac{\kappa\rho c}{T_0^2}\langle |\nabla T^*|^2 \rangle,$$

or

$$-\frac{\Delta T}{d}\frac{d\overline{T^*}}{dz}\bigg|_0 = \langle |\nabla T^*|^2 \rangle = \langle |\nabla T|^2 \rangle + \left\langle \left(\frac{d\overline{T^*}}{dz}\right)^2 \right\rangle.$$

Using (5) to eliminate $\overline{T^*}$ from this one readily reproduces (6). Thus (6) may be interpreted as expressing the over-all balance between the rate of generation of entropy in the layer as a consequence of heat conduction, and the difference between the entropy fluxes through the top and bottom surfaces which results from the fact that the heat fluxes are the same but the temperature is lower at the top. The relation (6) is called the 'second power integral' simply by analogy with (4); a physically more appropriate name would be 'entropy-flux integral'.

We now put these relations into dimensionless form by introducing d as length scale, ΔT as temperature scale and κ/d as velocity scale. The power integrals then become

$$R\langle wT\rangle = \langle|\nabla\mathbf{u}|^2\rangle, \tag{7}$$

$$\langle wT\rangle + \langle wT\rangle^2 - \langle\overline{wT}^2\rangle = \langle|\nabla T|^2\rangle, \tag{8}$$

where $R = \alpha g\Delta T d^3/\kappa\nu$ is the Rayleigh number and from now on all variables are taken to be dimensionless. The dimensionless negative mean temperature gradient $\beta\,\{= -(d/\Delta T)\,d\overline{T^*}/dz\}$ is given by the dimensionless form of (5)

$$\beta = 1 + \langle wT\rangle - \overline{wT}. \tag{9}$$

The value of β on the boundaries, $1+\langle wT\rangle$, is the dimensionless heat flux, i.e. the ratio of the actual heat flux to that which would occur with pure conduction. This is the *Nusselt number*

$$N = 1 + \langle wT\rangle. \tag{10}$$

Note that (10) and (7) imply that 'maximum heat flux' is equivalent to 'maximum viscous dissipation', ΔT being fixed.

The *critical Rayleigh number* R_c is defined as the greatest lower bound of the values of R for which an unstable infinitesimal perturbation of the purely conductive state exists. It is well known and easy to show that

$$R_c = \min\frac{\langle|\nabla\mathbf{u}|^2\rangle\langle|\nabla T|^2\rangle}{\langle wT\rangle^2},$$

the minimum being taken over all fields $\mathbf{u}$, T which vanish at the boundaries, satisfy $\nabla.\mathbf{u} = 0$, and satisfy the requirements of homogeneity. For the present problem R_c is about 1708. It is an immediate consequence of the power integrals that the only solution to the Boussinesq equations, the boundary conditions, and the requirements of homogeneity is pure conduction, if $R \leqslant R_c$. For suppose that we have a solution with $\mathbf{u} \not\equiv 0$. Then (7) implies that $\langle wT\rangle \neq 0$ (in fact $\langle wT\rangle > 0$; thus any statistically steady homogeneous convection must transport *more* heat than pure conduction) and so $R = \langle wT\rangle^{-1}\langle|\nabla\mathbf{u}|^2\rangle$. But from (8)

$$\begin{aligned}\langle wT\rangle &= \langle|\nabla T|^2\rangle + \langle\overline{wT}^2\rangle - \langle wT\rangle^2\\ &= \langle|\nabla T|^2\rangle + \langle(\overline{wT}-\langle wT\rangle)^2\rangle \geqslant \langle|\nabla T|^2\rangle.\end{aligned}$$

Therefore
$$R = \frac{\langle|\nabla\mathbf{u}|^2\rangle}{\langle wT\rangle^2}\langle wT\rangle > \frac{\langle|\nabla\mathbf{u}|^2\rangle\langle|\nabla T|^2\rangle}{\langle wT\rangle^2} \geqslant R_c.$$

We can now state our basic problem.

P1. *Given $R > R_c$, find the maximum value of $N = 1+\langle wT\rangle$ among all fields* $\mathbf{u}$, T *that vanish at $z = 0, 1$, satisfy the requirements of homogeneity, have $\nabla.\mathbf{u} = 0$, and satisfy the power integral relations* (7) *and* (8).

While the problem can be attacked in this form, it is equivalent to another variational problem which is free of integral constraints, and which seems to be somewhat more convenient to work with. We shall first state this equivalent problem, deduce some preliminary results about it, and then show its equivalence to P 1.

P 2. *Given $\lambda > 0$, find the minimum $m(\lambda)$ of the functional*

$$\mathscr{F} \equiv \mathscr{F}_\lambda(\mathbf{v},\theta) \equiv \langle\omega\theta\rangle^{-2}\{\langle\overline{\omega\theta}^2\rangle - \langle\omega\theta\rangle^2 + \lambda\langle|\nabla\mathbf{v}|^2\rangle\langle|\nabla\theta|^2\rangle\} \quad (\omega = \mathbf{k}.\mathbf{v})$$

among all fields $\mathbf{v}$, θ *that vanish at* $z = 0, 1$, *satisfy the requirements of homogeneity, and satisfy* $\nabla.\mathbf{v} = 0$.

We note that since $\langle\overline{\omega\theta}^2\rangle - \langle\omega\theta\rangle^2 = \langle(\overline{\omega\theta} - \langle\omega\theta\rangle)^2\rangle$, $\mathscr{F} > 0$ and that $\mathscr{F}$ is a homogeneous functional of degree 0. Suppose $0 < \lambda_1 < \lambda_2$, and let $\mathbf{v}_1$, θ_1, $\mathbf{v}_2$, θ_2 be minimizing functions corresponding to λ_1 and λ_2. Then we have

$$\begin{aligned}(\lambda_2-\lambda_1)\langle\omega_2\theta_2\rangle^{-2}\langle|\nabla\mathbf{v}_2|^2\rangle\langle|\nabla\theta_2|^2\rangle &= \mathscr{F}_{\lambda_2}(\mathbf{v}_2,\theta_2) - \mathscr{F}_{\lambda_1}(\mathbf{v}_2,\theta_2)\\ &\leqslant \mathscr{F}_{\lambda_2}(\mathbf{v}_2,\theta_2) - \mathscr{F}_{\lambda_1}(\mathbf{v}_1,\theta_1) \equiv m(\lambda_2) - m(\lambda_1)\\ &\leqslant \mathscr{F}_{\lambda_2}(\mathbf{v}_1,\theta_1) - \mathscr{F}_{\lambda_1}(\mathbf{v}_1,\theta_1) = (\lambda_2-\lambda_1)\langle\omega_1\theta_1\rangle^{-2}\langle|\nabla\mathbf{v}_1|^2\rangle\langle|\nabla\theta_1|^2\rangle.\end{aligned}$$

Dividing by $\lambda_2 - \lambda_1$ and letting $\lambda_2 \to \lambda_1$ we deduce

$$dm(\lambda)/d\lambda = \langle\omega\theta\rangle^{-2}\langle|\nabla\mathbf{v}|^2\rangle\langle|\nabla\theta|^2\rangle \geqslant R_c > 0,$$

and then using this result in the original chain of inequalities we get

$$(\lambda_2-\lambda_1)\,(dm/d\lambda)_2 \leqslant (\lambda_2-\lambda_1)\,(dm/d\lambda)_1,$$

i.e. m is an *increasing*, and $dm/d\lambda$ a *decreasing*, function of λ. Furthermore, it is clear that

$$\lim_{\lambda\to\infty}\frac{m(\lambda)}{\lambda} = \min\frac{\langle|\nabla\mathbf{v}|^2\rangle\langle|\nabla\theta|^2\rangle}{\langle\omega\theta\rangle^2} = R_c$$

and that consequently $dm/d\lambda$ decreases monotonically to R_c as $\lambda\to\infty$. To establish the equivalence of P 1 and P 2 we need to know also that $dm/d\lambda$ (or m/λ) increases without limit as $\lambda \to 0$. While this follows at once from the results of § 3, it can be shown directly by the following simple estimate.

We have

$$\theta^2 = \left(\int_0^z \theta_z dz'\right)^2 \leqslant \int_0^z \theta_z^2 dz'.\int_0^z dz' \leqslant z\int_0^1 \theta_z^2 dz',$$

and therefore
$$\overline{\theta^2} \leqslant z\langle\theta_z^2\rangle \leqslant z\langle|\nabla\theta|^2\rangle.$$

Similarly,
$$\overline{\omega^2} \leqslant z\langle|\nabla\omega|^2\rangle \leqslant z\langle|\nabla\mathbf{v}|^2\rangle.$$

Thus
$$|\overline{\omega\theta}| \leqslant \overline{\omega^2}^{\frac{1}{2}}\overline{\theta^2}^{\frac{1}{2}} \leqslant z\langle|\nabla\mathbf{v}|^2\rangle^{\frac{1}{2}}\langle|\nabla\theta|^2\rangle^{\frac{1}{2}}.$$

Let $A = \langle|\nabla\mathbf{v}|^2\rangle^{\frac{1}{2}}\langle|\nabla\theta|\rangle^{\frac{1}{2}}|\langle\omega\theta\rangle|^{-1}$. Then $|\overline{\omega\theta}| \leqslant |\langle\omega\theta\rangle|\,zA$, and

$$\langle(\overline{\omega\theta} - \langle\omega\theta\rangle)^2\rangle \geqslant \int_0^{A^{-1}}(|\langle\omega\theta\rangle| - |\overline{\omega\theta}|)^2 dz \geqslant \langle\omega\theta\rangle^2\int_0^{A^{-1}}(1-zA)^2 dz = \tfrac{1}{3}A^{-1}\langle\omega\theta\rangle^2.$$

Thus $\mathscr{F} \geqslant \frac{1}{3}A^{-1} + \lambda A^2$. As a function of A, the right-hand side has its minimum at $A = (6\lambda)^{-\frac{1}{3}}$, so

$$m(\lambda) \geqslant (\tfrac{3}{4}\lambda)^{\frac{1}{3}}. \qquad (11)$$

This shows that $m/\lambda \to \infty$ as $\lambda \to 0$. From these properties of $m(\lambda)$ it is clear that for any $R > R_c$ there is one and only one positive solution to the equation $m(\lambda) = R\lambda$.

The equivalence of P 1 and P 2 is embodied in the following statement:

For $R > R_c$, let λ (> 0) be the solution of $m(\lambda) = R\lambda$. Then any maximizing functions $\mathbf{u}$, T *for* P 1 *are multiples of minimizing functions* $\mathbf{v}$, θ *for* P 2, *and the maximum value of N is* $1 + m(\lambda)^{-1}$.

Proof. Because of the homogeneity of P 2, we shall show that maximizing functions $\mathbf{u}$, T are minimizing functions for $\mathscr{F}$, and then that given any minimizing functions $\mathbf{v}$, θ they can be renormalized to produce maximizing functions $\mathbf{u}$, T. First, suppose that $\mathbf{u}$, T satisfy the requirements of P 1. Then we have

$$m(\lambda) \leqslant \mathscr{F}_\lambda(\mathbf{u}, T) = \langle wT\rangle^{-2}\{\langle \overline{wT^2}\rangle - \langle wT\rangle^2 + R\lambda\langle wT\rangle[\langle wT\rangle + \langle wT\rangle^2 - \langle\overline{wT^2}\rangle]\}$$

$$= R\lambda + [\langle\overline{wT^2}\rangle - \langle wT\rangle^2]\langle wT\rangle^{-2}(1 - \langle wT\rangle R\lambda).$$

Since $\langle\overline{wT^2}\rangle - \langle wT\rangle^2 > 0$ this implies $1 - \langle wT\rangle R\lambda \geqslant 0$, and so

$$\langle wT\rangle \leqslant 1/R\lambda = 1/m(\lambda).$$

Thus N cannot exceed $1 + m(\lambda)^{-1}$. On the other hand, let $\mathbf{v}$, θ minimize $\mathscr{F}$, and set $\mathbf{u} = A\mathbf{v}$, $T = B\theta$ with

$$A = [\lambda\langle|\nabla\mathbf{v}|^2\rangle]^{-\frac{1}{2}}, \quad B = \frac{\langle|\nabla\mathbf{v}|^2\rangle}{R\langle\omega\theta\rangle}A. \tag{12}$$

One easily checks that $\mathbf{u}$ and T satisfy the power integral relations, and so are competitors for P 1, and $\langle wT\rangle = 1/R\lambda$. Thus $\mathbf{u}$ and T do maximize N, and being multiples of $\mathbf{v}$, θ of course also minimize $\mathscr{F}$.

The estimate (11) on $m(\lambda)$ already permits us to give a rough upper bound on N. For, given R, we have $R\lambda = m \geqslant (\frac{3}{4}\lambda)^{\frac{1}{3}}$. Therefore $\lambda^{-1} \leqslant (\frac{4}{3})^{\frac{1}{2}}R^{\frac{3}{2}}$. Since $N = 1 + 1/R\lambda$, we have

$$N \leqslant 1 + (\tfrac{4}{3}R)^{\frac{1}{2}}. \tag{13}$$

This estimate will be improved upon in the next section.

3. Maximum heat transport without continuity

It is clear from the derivation of (13) that any sufficiently strong estimate of $m(\lambda)$ from below will give an estimate of $N(R)$ from above. An obvious estimate of $m(\lambda)$ is obtained by minimizing the functional $\mathscr{F}$ over a larger class of fields $\mathbf{v}$, θ, by dropping the restriction $\nabla.\mathbf{v} = 0$. When this is done, it is clear from the form of $\mathscr{F}$ that the minimum can only be attained when $u = v = 0$, and we may as well start by replacing $\mathscr{F}$ by the functional

$$\mathscr{G}(\omega,\theta) = \langle\omega\theta\rangle^{-2}[\langle\overline{\omega\theta^2}\rangle - \langle\omega\theta\rangle^2 + \lambda\langle|\nabla\omega|^2\rangle\langle|\nabla\theta|^2\rangle]. \tag{14}$$

In the problem P 2, one of the requirements of homogeneity is that $\overline{\omega} = 0$. If this is retained in the problem of minimizing $\mathscr{G}$ it can be shown that the minimum is not attained, though it can be arbitrarily approximated. The same remark applies to the boundary conditions $\omega_z(0) = \omega_z(1) = 0$ which are satisfied by the solution to P 2 as a consequence of the continuity equation and the vanishing of the horizontal components of $\mathbf{v}$ on $z = 0, 1$. Thus it is simplest to drop these conditions from the start and formulate the problem:

P 3. *Given $\lambda > 0$, find the minimum $m_1(\lambda)$ of the functional $\mathscr{G}$ among all pairs of functions ω, θ which vanish on $z = 0$, 1 and for which all necessary horizontal averages exist.*

Evidently $m_1(\lambda) \leqslant m(\lambda)$, and by an argument almost identical with that given above to show the equivalence of P 1 and P 2 one can show that P 3 is similarly equivalent to the problem of maximizing N subject to the power integrals, the boundary conditions $\mathbf{u} = T = 0$ on $z = 0$, 1 and the requirements of homogeneity (without $\overline{w} = 0$), but without the requirement that $\nabla.\mathbf{u} = 0$.

We shall now solve P 3. The complete symmetry of the problem in ω and θ suggests that the minimum is attained with $\omega = \theta$, or because of the homogeneity, really with ω proportional to θ. The first step is to prove this.

The Euler equations for P 3 are

$$\lambda\langle|\nabla\theta|^2\rangle\nabla^2\omega+[(\mathscr{G}+1)\langle\omega\theta\rangle-\overline{\omega\theta}]\,\theta = 0 \tag{15}$$

and

$$\lambda\langle|\nabla\omega|^2\rangle\nabla^2\theta+[(\mathscr{G}+1)\langle\omega\theta\rangle-\overline{\omega\theta}]\,\omega = 0. \tag{16}$$

Because of the homogeneity we may clearly normalize so that

$$\lambda\langle|\nabla\theta|^2\rangle = \lambda\langle|\nabla\omega|^2\rangle = 1 \quad \text{and} \quad \langle\omega\theta\rangle > 0.$$

Supposing this done, and setting $\Phi(z) = (\mathscr{G}+1)\langle\omega\theta\rangle-\overline{\omega\theta}$, introduce as new dependent variables $\sigma = \frac{1}{2}(\omega+\theta)$, $\tau = \frac{1}{2}(\omega-\theta)$. Then the equations become

$$\nabla^2\sigma+\Phi\sigma = 0, \tag{17}$$

$$\nabla^2\tau-\Phi\tau = 0, \tag{18}$$

$$\Phi = A-\overline{\sigma^2}+\overline{\tau^2}, \tag{19}$$

where

$$A = (\mathscr{G}+1)\langle\sigma^2-\tau^2\rangle = (\mathscr{G}+1)\langle\omega\theta\rangle > 0.$$

We wish to show that a solution of these equations with $\sigma = \tau = 0$ on $z = 0$, 1 must have $\tau \equiv 0$. Now

$$\overline{\sigma\nabla^2\sigma} = \overline{\sigma\frac{\partial^2\sigma}{\partial z^2}}-\overline{(\sigma_x^2+\sigma_y^2)} = \frac{1}{2}\frac{d^2}{dz^2}\overline{\sigma^2}-\overline{|\nabla\sigma|^2} = -\Phi\overline{\sigma^2}$$

from (17), or

$$\tfrac{1}{2}d^2\overline{\sigma^2}/dz^2 = -\Phi\overline{\sigma^2}+\overline{|\nabla\sigma|^2}. \tag{20}$$

Similarly, using (18), we find

$$\tfrac{1}{2}d^2\overline{\tau^2}/dz^2 = +\Phi\overline{\tau^2}+\overline{|\nabla\tau|^2}. \tag{21}$$

Now (21) shows that $\tau \equiv 0$, if $\Phi \geqslant 0$ on $[0, 1]$, for $d\overline{\tau^2}/dz = 2\overline{\tau\tau_z} = 0$ on $z = 0, 1$, and integration of (21) from 0 to 1 would then give a contradiction unless $\tau \equiv 0$. Suppose then that $\Phi < 0$ on one or more intervals in $[0, 1]$. Since $\Phi > 0$ at 0 and 1 any such interval must be interior to $(0, 1)$. Let Φ be < 0 on (z_1, z_2), and zero at the end-points. Then at z_1,

$$0 \geqslant \frac{d\Phi}{dz} = -\frac{d\overline{\sigma^2}}{dz}+\frac{d\overline{\tau^2}}{dz},$$

i.e.

$$\left.\frac{d\overline{\sigma^2}}{dz}\right|_{z_1} \geqslant \left.\frac{d\overline{\tau^2}}{dz}\right|_{z_1}.$$

Similarly $$\left.\frac{d\overline{\tau^2}}{dz}\right|_{z_2} \geqslant \left.\frac{d\overline{\sigma^2}}{dz}\right|_{z_2}$$

Since $\Phi < 0$ on (z_1, z_2) and $A > 0$, (19) shows that $\overline{\sigma^2} > 0$ on (z_1, z_2) and integration of (20) over (z_1, z_2) then shows that

$$\left.\frac{d\overline{\sigma^2}}{dz}\right|_{z_2} > \left.\frac{d\overline{\sigma^2}}{dz}\right|_{z_1}$$

We thus have $$\left.\frac{d\overline{\tau^2}}{dz}\right|_{z_2} \geqslant \left.\frac{d\overline{\sigma^2}}{dz}\right|_{z_2} > \left.\frac{d\overline{\sigma^2}}{dz}\right|_{z_1} \geqslant \left.\frac{d\overline{\tau^2}}{dz}\right|_{z_1},$$

so that $d\overline{\tau^2}/dz$ must show a strictly positive increase over any interval like (z_1, z_2). On the other hand, (21) shows that $d\overline{\tau^2}/dz$ cannot decrease over any interval on which $\Phi \geqslant 0$, and thus if there are any intervals on which $\Phi < 0$, $d\overline{\tau^2}/dz$ must have a strictly positive increase from $z = 0$ to 1, which contradicts the fact that it is zero at both end-points. Thus there cannot be any intervals on which $\Phi < 0$, and the earlier argument then shows that $\tau \equiv 0$.

Thus in seeking the minimum of $\mathscr{G}(\omega, \theta)$ we need only consider the minimum of

$$\mathscr{G}_1(\phi) = \frac{\langle\overline{\phi^2}^2\rangle - \langle\phi^2\rangle^2 + \lambda\langle|\nabla\phi|^2\rangle^2}{\langle\phi^2\rangle^2}.$$

Let $f^2(z) = \overline{\phi^2}$. Then $|ff'| = |\overline{\phi\phi_z}| \leqslant \overline{\phi^2}^{\frac{1}{2}}\overline{\phi_z^2}^{\frac{1}{2}} \leqslant \overline{\phi^2}^{\frac{1}{2}}\,\overline{|\nabla\phi|^2}^{\frac{1}{2}}$, or $f'^2 \leqslant \overline{|\nabla\phi|^2}$. Thus

$$\mathscr{G}_1(\phi) \geqslant \mathscr{H}(f) \equiv \frac{\langle f^4\rangle - \langle f^2\rangle^2 + \lambda\langle f'^2\rangle^2}{\langle f^2\rangle^2},$$

and this inequality is an equality if ϕ is independent of x and y. The minimum of $\mathscr{H}$ thus gives the minimum of $\mathscr{G}_1$ and consequently also of $\mathscr{G}$. P 3 has been reduced to:

P 4. *For given $\lambda > 0$ find the minimum of $\mathscr{H}$ among functions $f(z)$ zero at $z = 0$ and* 1. (In $\mathscr{H}(f)$, the brackets $\langle\,\rangle$ mean simply an integration over $[0, 1]$, of course.) P 4 is readily solved; the Euler equation is

$$\lambda\langle f'^2\rangle f'' + (\mathscr{H}+1)\langle f^2\rangle f - f^3 = 0, \tag{22}$$

from which the general nature of the solution can be seen at once by regarding (22) as the equation of motion of a non-linear oscillator having a 'soft' spring. $[0, 1]$ must just cover a half-period, or an integral number of half-periods. It is convenient to normalize f so that the maximum amplitude of the oscillation is 1, and take $f'(0) > 0$. The 'energy integral' is then

$$\tfrac{1}{2}\lambda\langle f'^2\rangle + \tfrac{1}{2}(\mathscr{H}+1)\langle f^2\rangle(f^2-1) - \tfrac{1}{4}(f^4-1) = 0$$

or $$\lambda\langle f'^2\rangle f'^2 = [(\mathscr{H}+1)\langle f^2\rangle - \tfrac{1}{2}(1+f^2)][1-f^2]. \tag{23}$$

Let $k^2 = [2(\mathscr{H}+1)\langle f^2\rangle - 1]^{-1}$, so that (23) becomes

$$f'^2 = [2\lambda\langle f'^2\rangle k^2]^{-1}(1-k^2f^2)(1-f^2). \tag{24}$$

Thus from the origin out to the first maximum of f we have

$$z = [2\lambda\langle f'^2\rangle k^2]^{\frac{1}{2}}\int_0^f [(1-k^2f^2)(1-f^2)]^{-\frac{1}{2}}\,df. \tag{25}$$

Let n be the number of half-periods in $(0,1)$, and let $B^2 = 2\lambda\langle f'^2\rangle k^2$; we shall show presently that $n = 1$ for minimum $\mathscr{H}$, as would no doubt be expected. From (25) we obtain the following relations:

$$\frac{1}{2n} = B\int_0^1 [(1-k^2f^2)(1-f^2)]^{-\frac{1}{2}}\,df \equiv BK(k), \tag{26}$$

$$\begin{aligned}\langle f^2\rangle &= 2n\int_0^{(2n)^{-1}} f^2 dz = 2nB\int_0^1 [(1-k^2f^2)(1-f^2)]^{-\frac{1}{2}} f^2 df \\ &\equiv 2nBD(k),\end{aligned} \tag{27}$$

$$\langle f'^2\rangle = 2nB^{-1}\int_0^1 [(1-k^2f^2)(1-f^2)]^{\frac{1}{2}}\,df,$$

which, after some simple transformations, becomes

$$\langle f'^2\rangle = 2nB^{-1}[\tfrac{2}{3}K(k) - \tfrac{1}{3}(k^2+1)\,D(k)]. \tag{28}$$

Now from (26) and (27)

$$\langle f^2\rangle = D(k)/K(k) \tag{29}$$

and from (26) and (28)

$$\langle f'^2\rangle = 4n^2K(k)\,[\tfrac{2}{3}K(k) - \tfrac{1}{3}(k^2+1)\,D(k)]. \tag{30}$$

Then from (26), (30) and the definition of B we get

$$\lambda^{-1} = (2k^2/B^2)\langle f'^2\rangle = 32n^4K^3[\tfrac{2}{3}K - \tfrac{1}{3}(k^2+1)\,D]. \tag{31}$$

Similarly, from (29) and the definition of k we find

$$\mathscr{H} = -1 + (1/2\langle f^2\rangle)(1+k^{-2}) = (2k^2D)^{-1}[(1+k^2)\,K - 2k^2D]. \tag{32}$$

(31) and (32) together give a parametric representation of $\mathscr{H} = m_1(\lambda)$. Note that this relation has the form $\mathscr{H} = F(n^4\lambda)$ where F is independent of n. Since F is easily seen to be monotone increasing, the minimum $\mathscr{H}$ for a fixed λ is given by $n = 1$, as anticipated. $\lambda \to 0^+$ corresponds to $k \to 1^-$, and $\lambda \to \infty$ to $k \to 0^+$. From the known properties of the complete elliptic integrals K and D one can easily show that for small λ we have

$$m_1 \sim (\tfrac{64}{3}\lambda)^{\frac{1}{3}} \quad (\lambda \to 0) \tag{33}$$

and for large λ,

$$m_1 \sim \pi^4\lambda \quad (\lambda \to \infty). \tag{34}$$

The maximum Nusselt number N_1 permitted by the power integrals but without the continuity equation is $1 + [m_1(\lambda)]^{-1}$, where λ is related to R by $m_1(\lambda) = R\lambda$. From the above results a parametric representation of $N_1 = N_1(R)$ can be obtained and is

$$R = (16K^3/3D)\,[(1+k^2)\,K - 2k^2D]\,[2K - (1+k^2)\,D], \tag{35}$$

$$N_1 = (1+k^2)\,K[(1+k^2)\,K - 2k^2D]^{-1}. \tag{36}$$

For large R this is

$$N_1 \sim (\tfrac{3}{64}R)^{\frac{1}{2}}. \tag{37}$$

As $k \to 0$, $N_1 \to 1$ and $R \to \pi^4$, which in the present case of neglect of the continuity equation plays the role of R_c; $N_1 = 1$ for $R \leqslant \pi^4$, and then increases above 1 as R goes above π^4. A graph of $N_1(R)$ is shown in figure 1.

While (35) and (36) give a complete solution of P 4, and so of P 3, it is of interest to note that in the case of small λ (or large R) the problem can be solved more easily by a 'boundary-layer' method. Returning to equation (22), and anticipating that for $\lambda \to 0$, $\mathscr{H} \to 0$ and f is nearly constant except in thin boundary layers at $z = 0$ and $z = 1$, normalize so that $\langle f^2 \rangle = 1$. Then $f \to 1$ except in the boundary layers, which have thickness ϵ, say, and we thus have $\langle f'^2 \rangle = O(\epsilon^{-1})$.

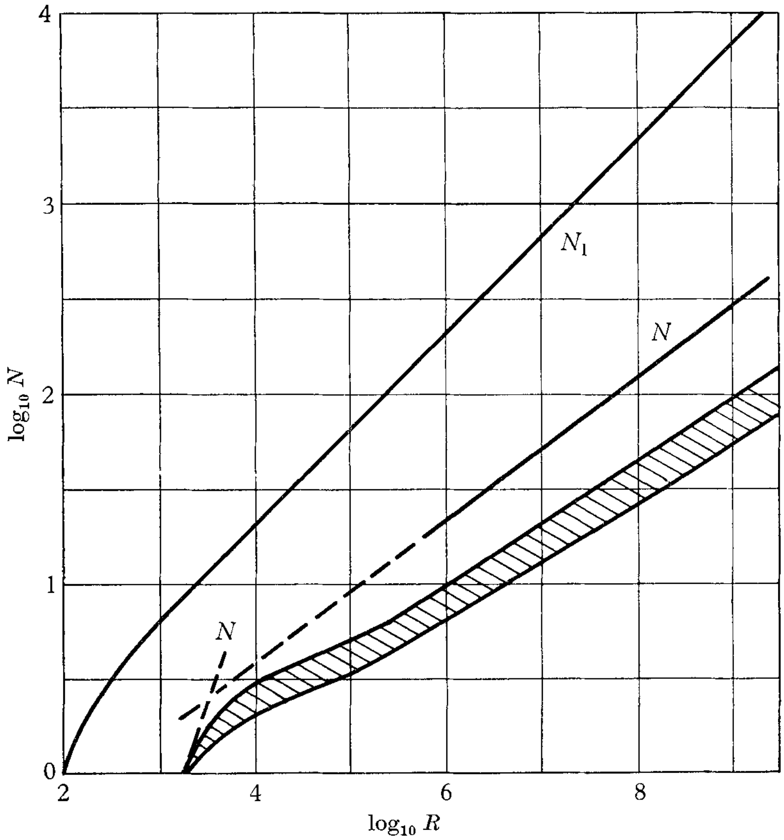

FIGURE 1. Heat flux. N_1, upper bound of §3. N, upper bound of §4. Shaded region: experiments.

Inside the boundary layers, $f'' = O(\epsilon^{-2})$; (22) then shows that we should take $\epsilon^3 = \lambda$. Considering only the boundary layer at $z = 0$, set $z = \epsilon\zeta$ and let $\lim \epsilon \langle f'^2 \rangle = 2C^2$. (22) then gives the boundary-layer equation

$$C^2\, d^2f/d\zeta^2 + f - f^3 = 0, \tag{38}$$

which is to be solved on $0 \leqslant \zeta < \infty$ with $f(0) = 0, f(\infty) = 1$. We have

$$2C^2 = \lim \epsilon \langle f'^2 \rangle = 2\int_0^\infty \left(\frac{df}{d\zeta}\right)^2 d\zeta, \tag{39}$$

the factor 2 on the right occurring because the contributions of both boundary layers must be counted in $\langle f'^2 \rangle$. The first integral of (38) is

$$C^2 (df/d\zeta)^2 + \tfrac{1}{2}(f^2 - 1) - \tfrac{1}{4}(f^4 - 1) = 0$$

or

$$C\, df/d\zeta = \tfrac{1}{2}(1 - f^2). \tag{40}$$

From (40) we find

$$C^2 = \int_0^\infty \left(\frac{df}{d\zeta}\right)^2 d\zeta = \int_0^1 \frac{df}{d\zeta} df = \frac{1}{2C}\int_0^1 (1-f^2)\,df = \frac{1}{3C};$$

thus $C = 3^{-\frac{1}{3}}$. Now

$$\langle f^4\rangle - \langle f^2\rangle^2 = \langle (f^2 - \langle f^2\rangle)^2\rangle = \langle (f^2-1)^2\rangle$$

$$= 2\epsilon \int_0^\infty (1-f^2)^2\, d\zeta = 4\epsilon C \int_0^\infty (1-f^2)\frac{df}{d\zeta}\, d\zeta = \tfrac{8}{3}\epsilon C.$$

Thus $\quad \mathscr{H} = \langle f^2\rangle^{-2}\,[\langle f^4\rangle - \langle f^2\rangle^2 + \lambda\langle f'^2\rangle^2] = \tfrac{8}{3}\epsilon C + \epsilon^3 (2C^2/\epsilon)^2 = 4\epsilon C,$

and therefore $m_1(\lambda) = 4(\frac{1}{3}\lambda)^{\frac{1}{3}} = (\frac{64}{3}\lambda)^{\frac{1}{3}}$, for small λ. This is of course the result (33) obtained previously. The function f is easily found also, and is

$$f = \tanh\left(\tfrac{1}{2}3^{\frac{1}{3}}\zeta\right).$$

4. Maximum heat transport with continuity

The ease with which the problem of the last section can be solved depends on two circumstances: first, the symmetry in ω and θ permits the reduction from two dependent variables to one, and secondly, the fact that the minimum can be shown to occur for a function independent of x and y permits the reduction to a single independent variable. When the continuity equation is retained, the symmetry is lost and the first simplification cannot be anticipated. However, the structure of P 2 is such that some simplification with regard to the independent variables is possible, though not so much as in P 3. The Euler equations for P 2 are obtained by setting the variation of $\mathscr{F} - 2\langle p\nabla.\mathbf{v}\rangle$ equal to zero; $p = p(x,y,z)$ is a Lagrange multiplier function introduced to take account of the constraint $\nabla.\mathbf{v} = 0$. This gives

$$\lambda\langle|\nabla\mathbf{v}|^2\rangle\nabla^2\theta + [(\mathscr{F}+1)\langle\omega\theta\rangle - \overline{\omega\theta}]\,\omega = 0, \tag{41}$$

$$\lambda\langle|\nabla\theta|^2\rangle\nabla^2\mathbf{v} + [(\mathscr{F}+1)\langle\omega\theta\rangle - \overline{\omega\theta}]\,\theta\mathbf{k} - \langle\omega\theta\rangle^2\nabla p = 0. \tag{42}$$

As in the linear stability problem for convection it is possible to obtain a pair of equations for ω and θ by taking (41) and the z-component of the double curl of (42), the latter being

$$\lambda\langle|\nabla\theta|^2\rangle\nabla^4\omega + [(\mathscr{F}+1)\langle\omega\theta\rangle - \overline{\omega\theta}]\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\theta = 0. \tag{43}$$

Now (41) and (43) are non-linear equations, but the non-linearity is of a specially simple form in that the equations do admit solutions which are eigenfunctions of the horizontal Laplacian $\Delta_1 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, i.e. depend on x and y only through a factor of (essentially) the form $\sin a_1 x \sin a_2 y$. Now suppose we find a solution of this form, corresponding to a particular eigenvalue $-a^2$ of Δ_1 ($a = [a_1^2 + a_2^2]^{\frac{1}{2}}$ is the total horizontal wave-number); since the Euler equations are satisfied, any small variation about this solution (and this variation need not have horizontal wave-number a, or any single wave-number) must give, to first order, zero variation in $\mathscr{F}$. However, such a solution might not give a minimum of $\mathscr{F}$; we might consider minimizing $\mathscr{F}$ by restricting the competition to functions of wave-number a, and it is easy to see that the solution of this problem would be a solution of the Euler equations for the full problem, but since presumably the minimum of $\mathscr{F}$ among functions of horizontal wave-number a is dependent on the value of a chosen, most of these could not be minima of $\mathscr{F}$ among all functions. Note that a small change in a does not give a small change in the *functions* uniformly over the (x, y)-plane, even though it presumably corresponds to a small change in the *vertical* dependence of the functions which minimize for fixed a. Thus the first derivative of this minimum $\mathscr{F}$ with respect to a need not be zero, though the Euler equations are satisfied. But if we choose that value of a (presumably unique) which gives the *least* minimum of $\mathscr{F}$ among functions of fixed horizontal wave-number, we shall have a plausible candidate for the absolute minimum of $\mathscr{F}$. Unfortunately this is really only the minimum of $\mathscr{F}$ among functions with a *single*, but unspecified, horizontal wave-number, and it is conceivable that there might be a *lower* minimum achieved by functions which consist of a mixture of different horizontal wave-numbers. This latter possibility seems unlikely to me, but I have not been able to prove that it does not occur. Lacking such a proof, and having been unable to develop effective methods for attacking the problem when a mixture of different horizontal wave-numbers are present, we must proceed on the basis of the conjecture that the minimum $\mathscr{F}$ is achieved for functions with a single horizontal wave-number. Thus we shall actually only solve P 2 subject to this additional restriction. If the conjecture is wrong, we shall not have obtained the true minimum; nevertheless, we shall in any case have estimates on the degree of success which the present general approach can have. The method of maximizing heat transport subject to the power integrals and continuity cannot limit the Nusselt number any more strongly than the limit given by the results of the present section, but it certainly limits it at least as strongly as the limit obtained in § 3. It is not implied here of course that the *real* flow has only a single horizontal wave-number; the Boussinesq equations would not permit that. It is only the solution to the mathematical problem P 1 or P 2 that is conjectured to be of this type, and in fact all that is needed is that among those fields which minimize $\mathscr{F}$ there should be *one* with a single horizontal wave-number.

Since we wish to formulate the problem entirely in terms of ω and θ, we must express $\langle|\nabla \mathbf{v}|^2\rangle$ in terms of ω. Now if ω is given, the horizontal components μ, ν of $\mathbf{v}$ are not uniquely determined by $\nabla . \mathbf{v} = 0$, nor in fact is $\langle|\nabla \mathbf{v}|^2\rangle$; however, given ω and the assumption that only one horizontal wave-number is present, there is a unique minimum value of $\langle|\nabla \mathbf{v}|^2\rangle$ and this is of course all that is needed. In fact we have (letting ∇_1 be the horizontal gradient operator):

$$\begin{aligned}\langle|\nabla \mathbf{v}|^2\rangle &= \langle|\nabla\omega|^2\rangle + \langle|\nabla\mu|^2\rangle + \langle|\nabla\nu|^2\rangle \\ &= \langle\omega_z^2\rangle + \langle|\nabla_1\omega|^2\rangle + \langle\mu_z^2\rangle + \langle|\nabla_1\mu|^2\rangle + \langle\nu_z^2\rangle + \langle|\nabla_1\nu|^2\rangle.\end{aligned}$$

Now from the continuity equation, $\omega_z = -\mu_x - \nu_y$, so

$$\begin{aligned}\langle\omega_{zz}^2\rangle = \langle(\mu_{xz}+\nu_{yz})^2\rangle &\equiv \langle\mu_{zx}^2+\mu_{zy}^2\rangle + \langle\nu_{zx}^2+\nu_{zy}^2\rangle \\ &\quad - \langle(\mu_{yz}-\nu_{xz})^2\rangle + 2\langle\mu_{xz}\nu_{yz}\rangle - 2\langle\mu_{yz}\nu_{xz}\rangle.\end{aligned}$$

On using the horizontal homogeneity, the last two terms are seen to cancel, and $\langle \mu_{zx}^2 + \mu_{zy}^2 \rangle = -\langle \mu_z \nabla_1^2 \mu_z \rangle = a^2 \langle \mu_z^2 \rangle$; similarly $\langle \nu_{zx}^2 + \nu_{zy}^2 \rangle = a^2 \langle \nu_z^2 \rangle$. Thus

$$\langle \mu_z^2 \rangle + \langle \nu_z^2 \rangle = a^2 \langle \omega_{zz}^2 \rangle + a^{-2} \langle \Omega_z^2 \rangle,$$

where Ω is the vertical component of the vorticity of the horizontal part of $\mathbf{v}$. Also

$$\begin{aligned} \langle |\nabla_1 \mu|^2 \rangle + \langle |\nabla_1 \nu|^2 \rangle &= \langle \mu_x^2 + \mu_y^2 + \nu_x^2 + \nu_y^2 \rangle \\ &= \langle (\mu_x + \nu_y)^2 \rangle + \langle (\mu_y - \nu_x)^2 \rangle - 2\langle \mu_x \nu_y \rangle + 2\langle \mu_y \nu_x \rangle. \end{aligned}$$

Again, the last two terms cancel, and from the continuity equation we find $\langle |\nabla_1 \mu|^2 \rangle + \langle |\nabla_1 \nu|^2 \rangle = \langle \omega_z^2 \rangle + \langle \Omega^2 \rangle$. Combining these results we get, since

$$\langle |\nabla_1 \omega|^2 \rangle = -\langle \omega \nabla_1^2 \omega \rangle = a^2 \langle \omega^2 \rangle,$$

$$\langle |\nabla \mathbf{v}|^2 \rangle = \langle a^{-2} \omega_{zz}^2 + 2\omega_z^2 + a^2 \omega^2 \rangle + \langle a^{-2} \Omega_z^2 + \Omega^2 \rangle.$$

Thus the minimum $\langle |\nabla \mathbf{v}|^2 \rangle$ for a prescribed ω is attained when the horizontal part of the flow has no vertical component of vorticity, and it is easily seen that this condition can always be realized for any ω; for instance, if

$$\omega = 2\omega(z) \sin a_1 x \sin a_2 y$$

we choose

$$\mu = \{(1+k)/a_1\}\, \omega'(z) \cos a_1 x \sin a_2 y,$$
$$\nu = \{(1-k)/a_2\}\, \omega'(z) \sin a_1 x \cos a_2 y,$$

with $k(a_1^2 + a_2^2) = a_1^2 - a_2^2$. We may thus simply take

$$\langle |\nabla \mathbf{v}|^2 \rangle = \langle a^{-2} \omega_{zz}^2 + 2\omega_z^2 + a^2 \omega^2 \rangle \tag{44}$$

and leave μ and ν out of consideration, remembering only to satisfy $\Omega = 0$ if and when they are ultimately determined.

We now take

$$\omega = \omega(z)\, \phi(x,y), \tag{45}$$

$$\theta = \theta(z)\, \phi(x,y), \tag{46}$$

where ϕ is some eigenfunction of the horizontal Laplacian with mean-square value 1, for instance, $\phi = 2 \sin a_1 x \sin a_2 y$, $a_1^2 + a_2^2 = a^2$. It is clear that the same ϕ should be used for θ as for ω, for any part of θ which is horizontally uncorrelated with ω will make no contribution to $\omega\theta$, but will increase $\langle |\nabla \theta|^2 \rangle$. Our problem is now to minimize the functional

$$\mathscr{F} = \langle \omega\theta \rangle^{-2} \{ \langle \omega^2 \theta^2 \rangle - \langle \omega\theta \rangle^2 + \lambda \langle \theta'^2 + a^2 \theta^2 \rangle \langle a^{-2} \omega''^2 + 2\omega'^2 + a^2 \omega^2 \rangle \}, \tag{47}$$

where ω and θ are from now on functions of z alone, and the minimum is to be sought among functions satisfying $\omega = \omega' = \theta = 0$ on $z = 0, 1$; we must also choose a appropriately, so as to minimize $\mathscr{F}$.

Even in this simplified form, this is a difficult problem, and we shall attack it only in the case of small λ (large R), making use of a boundary-layer method similar to that used at the end of § 3. In the previous case, $\langle a^{-2} \omega''^2 + 2\omega'^2 + a^2 \omega^2 \rangle$ was replaced by $\langle \omega'^2 + a^2 \omega^2 \rangle$ and $a = 0$ was obviously optimal. In the present case this is not so; continuity prevents the use of very long wavelengths, and the problem of determining the correct boundary-layer thickness is complicated by the need to determine $a(\lambda)$ simultaneously.

It is natural to start by normalizing ω and θ so that $\langle\omega\theta\rangle = 1$ and so that ω and $\theta \to 1$ away from the boundary, just as we did in § 3. It is clear from the form of $\mathscr{F}$ (see (47)), that to minimize $\mathscr{F}$ we shall need to have $\omega\theta$, which is zero at $z = 0$, grow rapidly to the value 1, the rapidity of this growth being restrained by the need to prevent the dissipation integrals from being too large. However, because of the lack of symmetry, we must not expect ω and θ necessarily to grow in the same way; indeed the higher derivatives occurring in the ω dissipation integral suggest that it may be preferable to have ω vary rather slowly, and allow θ to overshoot its limiting value 1 so as to achieve $\omega\theta \simeq 1$ before ω has increased much, $\omega\theta$ thereafter remaining nearly 1 as ω continues to increase to the value 1 far from the boundary. The increase in $\langle|\nabla\theta|^2\rangle$ produced by this excessive variation in θ may be more than offset by the smaller value of $\langle|\nabla\mathbf{v}|^2\rangle$ which it permits, without increasing $\langle(\omega\theta-1)^2\rangle$. If this should be the case, we may anticipate that θ may be larger than $O(1)$ (in λ) inside the region in which $\omega\theta$ differs appreciably from 1, which we shall call the 'boundary layer', though as has been suggested this is not necessarily the 'boundary layer' in ω and θ separately, but only in their product. To allow for this possibility, we shall set

$$\omega = \omega_1\lambda^p, \quad \theta = \theta_1\lambda^{-p}, \quad z = \zeta\lambda^r, \quad a^2 = b^2\lambda^{-q},$$

supposing that b is of order 1 and ω_1 and θ_1 are of order 1 inside the boundary layer. With these variables, and anticipating that despite the possible large peak of θ in the boundary layer we shall still have $\langle\omega^2\rangle = \langle\theta^2\rangle = 1$ (within the boundary-layer approximation) we find for $\mathscr{F}$, keeping in mind that there is also a boundary layer at $z = 1$,

$$\mathscr{F} = 2\lambda^r\int_0^\infty (1-\omega_1\theta_1)\,d\zeta + \lambda\left[2\lambda^{-2p-r}\int_0^\infty \left(\frac{d\theta_1}{d\zeta}\right)^2 d\zeta + b^2\lambda^{-q}\right]$$
$$\times\left[2b^{-2}\lambda^{q+2p-3r}\int_0^\infty\left(\frac{d^2\omega_1}{d\zeta^2}\right)^2 d\zeta + 4\lambda^{2p-r}\int_0^\infty\left(\frac{d\omega_1}{d\zeta}\right)^2 d\zeta + b^2\lambda^{-q}\right]. \quad (48)$$

We must now choose the exponents p, q, r so as to minimize $\mathscr{F}$ for small λ, i.e. we must maximize the minimum exponent of λ which occurs in $\mathscr{F}$. These exponents are:

$$r, \quad 1+q-4r, \quad 1-2r, \quad 1-2p-r-q, \quad 1+2p-3r, \quad 1-q+2p-r, \quad 1-2q.$$

Let e be the minimum exponent. Then, among others, the following inequalities must hold:

$$r \geqslant e, \quad (49)$$
$$1+q-4r \geqslant e, \quad (50)$$
$$1-2q \geqslant e. \quad (51)$$

Multiplying (49) by 8, (50) by 2, and adding to (51) we get $3 \geqslant 11e$, so $e \leqslant \frac{3}{11}$ in any case. Suppose $e = \frac{3}{11}$. Then by adding 4 times (49) to (50) and using (51) we get $\frac{4}{11} = 5e-1 \leqslant q \leqslant \frac{1}{2}(1-e) = \frac{4}{11}$, hence $q = \frac{4}{11}$. Similarly, from (49) and (50) we get

$$\tfrac{3}{11} = e \leqslant r \leqslant \tfrac{1}{4}(1+q-e) = \tfrac{3}{11},$$

so $r = \frac{3}{11}$. From the fourth and fifth of the original exponents one then finds $\frac{1}{11} = e-1+3r \leqslant 2p \leqslant 1-r-q-e = \frac{1}{11}$, so $p = \frac{1}{22}$. If in fact we set $p = \frac{1}{22}$,

$q = \frac{4}{11}$, $r = \frac{3}{11}$ we find that all the orignal exponents take the maximum possible value $\frac{3}{11}$, except the third and sixth, which become $\frac{5}{11}$. Thus the maximum $\frac{3}{11}$ *can* be obtained, and in only one way. With the above choice of exponents we obtain $\mathscr{F} = O(\lambda^{\frac{3}{11}})$, and setting $\mathscr{F} = \mathscr{F}_1\lambda^{\frac{3}{11}}$ we get, within the boundary layer approximation,

$$\mathscr{F}_1 = 2\int_0^\infty (1-\omega_1\theta_1)^2 d\zeta + \left[2\int_0^\infty \left(\frac{d\theta_1}{d\zeta}\right)^2 d\zeta + b^2\right]\left[2b^{-2}\int_0^\infty \left(\frac{d^2\omega_1}{d\zeta^2}\right)^2 d\zeta + b^2\right]. \quad (52)$$

$\mathscr{F}_1$ is to be minimized by choice of b and the functions θ_1 and ω_1, subject to the boundary conditions

$$\omega_1(0) = \omega_1'(0) = \theta_1(0) = 0,$$

$$\theta_1 \to 0, \quad \omega_1\theta_1 \to 1 \quad \text{as} \quad \zeta \to \infty.$$

Varying ω_1, θ_1, and b^2 we find the following equations which must be satisfied for a minimum

$$b^{-2}\left[2\int_0^\infty \theta_1'^2 d\zeta + b^2\right]\frac{d^4\omega_1}{d\zeta^4} - (1-\omega_1\theta_1)\theta_1 = 0, \quad (53)$$

$$\left[2b^{-2}\int_0^\infty \omega_1''^2 d\zeta + b^2\right]\frac{d^2\theta_1}{d\zeta^2} + (1-\omega_1\theta_1)\omega_1 = 0, \quad (54)$$

$$\frac{2}{b^2}\int_0^\infty \omega_1''^2 d\zeta + b^2 + \left[2\int_0^\infty \theta_1'^2 d\zeta + b^2\right]\left[-2b^{-4}\int_0^\infty \omega_1''^2 d\zeta + 1\right] = 0. \quad (55)$$

From (53) and (54) we obtain

$$b^{-2}\left[2\int_0^\infty \theta_1'^2 d\zeta + b^2\right]\int_0^\infty \omega_1''^2 d\zeta = \int_0^\infty (1-\omega_1\theta_1)\omega_1\theta_1 d\zeta$$

$$= \left[2b^{-2}\int_0^\infty \omega_1''^2 d\zeta + b^2\right]\int_0^\infty \theta_1'^2 d\zeta$$

and thus
$$\int_0^\infty \omega_1''^2 d\zeta = b^2\int_0^\infty \theta_1'^2 d\zeta \equiv b^2\mu, \quad \text{say.}$$

Putting these in (55) we find that

$$2\mu + b^2 + (2\mu + b^2)(-2\mu b^{-2} + 1) \equiv 2(2\mu + b^2)(1 - \mu b^{-2}) = 0$$

and thus $\mu = b^2$. Using this in (53) and (54) we get

$$3(d^4\omega_1/d\zeta^4) - (1-\omega_1\theta_1)\theta_1 = 0, \quad (56)$$

$$3b^2(d^2\theta_1/d\zeta^2) + (1-\omega_1\theta_1)\omega_1 = 0. \quad (57)$$

Setting
$$\omega_1 = (3b^4)^{\frac{1}{6}}\Omega, \quad \theta_1 = (3b^4)^{-\frac{1}{6}}\Theta, \quad \zeta = (3b)^{\frac{1}{3}}\xi, \quad (58)$$

(56) and (57) become

$$d^4\Omega/d\xi^4 - (1-\Omega\Theta)\Theta = 0, \quad (59)$$

$$d^2\Theta/d\xi^2 + (1-\Omega\Theta)\Omega = 0. \quad (60)$$

These equations, with the boundary conditions $\Omega(0) = \Omega'(0) = 0 = \Theta(0)$, $\Theta \to 0$ and $\Omega\Theta \to 1$ as $\xi \to \infty$, determine Ω and Θ, without requiring a knowledge of b. However since

$$b^2 = \mu = \int_0^\infty \left(\frac{d\theta_1}{d\zeta}\right)^2 d\zeta = (3b^4)^{-\frac{1}{3}}(3b)^{-\frac{1}{3}}\int_0^\infty \left(\frac{d\Theta}{d\xi}\right)^2 d\xi,$$

once Θ is known we can determine b

$$b^{\frac{11}{3}} = 3^{-\frac{2}{3}} \int_0^\infty \left(\frac{d\Theta}{d\xi}\right)^2 d\xi. \tag{61}$$

It is easy to see from (59) and (60) that

$$\int_0^\infty \Omega''^2 d\xi = \int_0^\infty \Theta'^2 d\xi.$$

Also, (59) and (60) have the following first integral

$$\Theta'^2 - 2\Omega'\Omega''' + \Omega''^2 = (1-\Omega\Theta)^2, \tag{62}$$

from which follows:

$$\int_0^\infty (1-\Omega\Theta)^2 d\xi = \int_0^\infty \theta'^2 d\xi + 3\int_0^\infty \Omega''^2 d\xi = 4\int_0^\infty \Theta'^2 d\xi.$$

Using (58), the terms in $\mathscr{F}_1$ (see (52)) can be expressed in terms of b and these integrals, and so in terms of b alone. This gives

$$\mathscr{F}_1 = 33b^4. \tag{63}$$

It is of interest to note that (59) and (60) are the Euler equations characterizing the minimum of the functional

$$\mathscr{I} = \frac{1}{6}\int_0^\infty [\Theta'^2 + \Omega''^2 + (1-\Omega\Theta)^2]\, d\xi \tag{64}$$

(with the same boundary conditions as before) and that the minimum value σ of $\mathscr{I}$ is $3^{\frac{2}{3}} b^{\frac{11}{3}}$. This gives a simple way of estimating b by using trial functions for Ω and Θ in the integral (64).

With the minimum value (63) for $\mathscr{F}_1$ we have determined the minimum $m(\lambda)$ for $\mathscr{F}$, in the case of small λ

$$m(\lambda) = 33b^4\lambda^{\frac{3}{11}}. \tag{65}$$

The relation between λ and R being $m(\lambda) = R\lambda$ we have

$$R = 33b^4\lambda^{-\frac{8}{11}}, \quad \lambda = b^{\frac{11}{2}}(R/33)^{-\frac{11}{8}}. \tag{66}$$

Within the boundary-layer approximation, the Nusselt number is

$$N \equiv 1 + m(\lambda)^{-1} \cong (R\lambda)^{-1};$$

thus

$$N = \tfrac{1}{33} b^{-\frac{11}{2}} (R/33)^{\frac{3}{8}} \equiv \tfrac{1}{11}\sigma^{-\frac{3}{2}}(R/33)^{\frac{3}{8}}, \tag{67}$$

where $\sigma = 3^{\frac{2}{3}} b^{\frac{11}{3}}$ is the minimum of the functional $\mathscr{I}$ of (64). The functions $w(z)$, $T(z)$, etc., can now be expressed in terms of $\Omega(\xi)$ and $\Theta(\xi)$ by tracing back through the various renormalizations. The results are

$$w(z) = \sigma^{-\frac{1}{2}}(R/33)^{\frac{3}{8}}\,\Omega(\xi), \tag{68}$$

$$T(z) = (11\sigma)^{-1}\,\Theta(\xi), \tag{69}$$

$$\beta(z) = \tfrac{1}{11}\sigma^{-\frac{3}{2}}(R/33)^{\frac{3}{8}}[1-\Omega\Theta], \tag{70}$$

$$z = \sigma^{\frac{1}{2}}(R/33)^{-\frac{3}{8}}\,\xi, \tag{71}$$

$$a = (R/33)^{\frac{1}{4}}. \tag{72}$$

This completes the boundary-layer solution of the problem, except for the numerical determination of b (or σ) and the functions Ω and Θ. Before presenting the numerical results we return briefly to the consideration of $m(\lambda)$ for large λ. It was shown in § 2 that for large λ, $m(\lambda) \sim R_c\lambda$, and so that $N(R)$ is reduced to 1 as $R \to R_c^+$. It is not difficult to estimate $m(\lambda)$ a little more precisely, and so obtain $dN/dR|_{R=R_c^+}$. For the minimizing functions we have

$$\lambda^{-1}[m(\lambda) - R_c\lambda] = \lambda^{-1}\langle\omega\theta\rangle^{-2}[\langle\overline{\omega\theta^2}\rangle - \langle\omega\theta\rangle^2] + [\langle\omega\theta\rangle^{-2}\langle|\nabla \mathbf{v}|^2\rangle\langle|\nabla\theta|^2\rangle - R_c]. \quad (73)$$

For $\lambda^{-1} \to 0$ the minimizing functions differ from their limiting values by terms of order λ^{-1}, but the second term on the right of (73) is zero to a higher order, because of the variational definition of R_c. Thus if ω_0 and θ_0 are minimizing functions for $\lambda \to \infty$ (namely, proportional to the solution functions w and T of the linear stability problem) we have

$$\begin{aligned}\lambda^{-1}[m(\lambda) - R_c\lambda] &= \lambda^{-1}\langle\omega_0\theta_0\rangle^{-2}[\langle\overline{\omega_0\theta_0^2}\rangle - \langle\omega_0\theta_0\rangle^{-2}] + o(\lambda^{-1}) \\ &= J\lambda^{-1} + o(\lambda^{-1}), \quad \text{say.}\end{aligned} \quad (74)$$

Thus for large λ, $m(\lambda) \sim R_c\lambda + J$, and this gives, for R slightly greater than R_c,

$$N \sim 1 + J^{-1}(R - R_c)/R_c. \quad (75)$$

The determination of J requires a knowledge of ω_0 and θ_0, and as solution functions for the linear stability problem they have been computed for the present case of rigid boundaries, by Pellew & Southwell (1940) and others; a convenient reference is Chandrasekhar (1961), Ch. II, table II. Chandrasekhar has in fact given (1961, Appendix I) the values of the integrals needed, and we find $J \simeq 0{\cdot}6919$. In (67) and (75) we have the asymptotic behaviour at the two ends of the $N = N(R)$ curve. A preliminary value of the constant σ in (67) can be obtained by using trial functions in (64); a convenient choice is

$$\Theta = A(c\xi)^{-\frac{1}{3}}(1 - e^{-(c\xi)^{\frac{4}{3}}}), \quad \Omega = A^{-1}(c\xi)^{\frac{1}{3}}(1 - e^{-(c\xi)^{\frac{4}{3}}}).$$

When A and c are adjusted to minimize $\mathscr{I}$, one finds $0{\cdot}35$ for an upper estimate on σ. The numerical calculation to be mentioned in the next section gives $\sigma \simeq 0{\cdot}337$, and for the upper bound (67) on the Nusselt number, for large Rayleigh number, this corresponds to $N \simeq (R/248)^{\frac{3}{8}}$. Figure 1 shows the asymptotic forms (67) and (75), as well as the upper bound $N_1(R)$ obtained in § 3. The experimental data show a certain amount of scatter, but seem to lie in the shaded region shown in figure 1. In drawing this shaded region, account has been taken of the experiments of Silveston and Mull & Reiher (summarized in Chandrasekhar's book 1961, § 18), of Malkus (1954*b*), and of the recent work of Globe & Dropkin (1959). Silveston's rather precise experiments near $R = R_c$ give a value of $d(\log R)/d(\log N)$ about $\frac{2}{3}$ of that given by (75). At the high end of the experimentally studied range (about $R = 10^9$), the upper bound (67) is about four times the observed Nusselt number, while the upper bound N_1 obtained without consideration of the continuity equation is about 100 times the observed value.

5. Discussion

The exponent $\frac{3}{8}$ in the asymptotic expression (67) for the Nusselt number deserves some comment, since it is sometimes stated that at large Rayleigh number the Nusselt number should vary as $R^{\frac{1}{3}}$. The argument for the exponent $\frac{1}{3}$ is briefly the following: large Rayleigh number can be achieved by making the plate separation d large, keeping ΔT fixed. For $d \to \infty$, then, the heat flux $(\kappa \Delta T/d)\, N$ should become independent of d, this being the heat flux into a semi-infinite region from a heated lower surface. Since dimensional considerations show that N can depend only on R and the Prandtl number, and of these only R depends on d (being proportional to d^3) N must vary as $R^{\frac{1}{3}}$ for large R to make the heat flux independent of d, as $R \to \infty$. This argument would be conclusive if it were known that a statistically steady convection, with a finite heat flux, into a semi-infinite region exists, but this is not known; that it is not physically obvious is clear from the fact that a steady *conduction* into a semi-infinite region does not in fact exist. Thus $N \propto R^{\frac{1}{3}}$ can only be regarded as a more or less plausible conjecture, and it is not impossible that the real heat flux might, like the upper bound obtained above, vary as $R^{\frac{3}{8}}$. The difference between $\frac{3}{8}$ and $\frac{1}{3}$ is also sufficiently small that a definitive experimental decision between the two would require quite precise measurements over a wide range of very large Rayleigh numbers, an experimental programme with many inherent difficulties. On the whole, however, the experiments seem in somewhat better agreement with $\frac{1}{3}$ than $\frac{3}{8}$, and even if the exponent $\frac{3}{8}$ is fitted to the experimental data, the proportionality constant is less than that of the maximizing flow, as is clear from figure 1. The imposition of additional constraints will of course reduce the upper bound, and might well alter the exponent; to continue with the programme of this paper, such constraints should be obtained as consequences of the equations of motion, but it is perhaps of some interest to note here that Malkus's requirement of a non-positive mean temperature gradient (which was not derived from the equations of motion but suggested on the basis of somewhat dubious physical considerations) is not strong enough to alter the exponent. Malkus's requirement is not in fact satisfied by the maximizing fields, as we shall see in detail presently, but it can be shown that even if it is imposed, only the proportionality constant, not the exponent, can be changed.

It has recently been suggested by Kraichnan (1962), in connexion with his modified 'mixing-length' investigation of turbulent convection, that at very large Rayleigh numbers, and for Prandtl numbers of the order of 1 or smaller, the Nusselt number should vary as $R^{\frac{1}{2}}(\log R)^{-\frac{3}{2}}$, a more rapid rate of growth than $R^{\frac{3}{8}}$. If this is correct, then the conjecture made above in § 4 that the maximizing flow has (or may be taken to have) only a single horizontal wavenumber, must be incorrect. It is only of course N_1 which has been rigorously proved to be an upper bound, and N_1 does exceed Kraichnan's value; however, the conjecture of § 4 will be retained in this paper.

The explicit determination of the functions Ω and Θ from the differential equations (59) and (60) seems to be rather more difficult than in the analogous case treated in § 3. After a reasonably diligent but unsuccessful search for a closed-form solution, a numerical calculation was carried out, the final computation being made by the following method: the first integral (62) together with the boundary conditions shows that $\Theta'^2(0)+\Omega''^2(0) = 1$, and consequently the solution desired is to be found among the two-parameter family of solutions characterized by the initial conditions: $\Theta(0) = \Omega(0) = \Omega'(0) = 0$, $\Theta'(0) = \alpha$, $\Omega''(0) = (1-\alpha^2)^{\frac{1}{2}}$, $\Omega'''(0) = \beta$. For most values of α and β, such a solution diverges as $\xi \to \infty$ and the value of the functional $\mathscr{I}$ (cf. (64)) for it is infinite. It is only when a certain relation between α and β holds that $\mathscr{I}$ is finite, and for most of these solutions Θ approaches a non-zero constant as $\xi \to \infty$. We wish to find that solution for which $\Theta \to 0$ at infinity. Since it is known from the use of trial functions that $\sigma < 0{\cdot}35$, the equations were integrated from $\xi = 0$, for different values of α and β, the value of the partial integral $\mathscr{I}_0^\xi$ being computed as the integration proceeded and integration being stopped as soon as $\mathscr{I}_0^\xi$ exceeded 0·35. In this way it was possible to determine with considerable accuracy the value of β for a given α for which a solution with finite $\mathscr{I}$ exists, because the divergent solutions soon begin to grow very rapidly. It was found possible to carry this integration out to about $\xi = 5$ before the accumulation of errors became troublesome, and the solutions for which $\Theta \to$ constant could thus be determined quite accurately to about $\xi = 4$. To find which of these solutions has $\Theta \to 0$, the solutions computed as above were matched up with an asymptotic solution, the matching being made at about $\xi = 3$. The asymptotic solution is obtained as follows: since Θ and $1-\Omega\Theta$ approach zero, (59) shows that for large ξ, Ω must be essentially a polynomial of degree at most 3; actually since $\Omega'' \to 0$ as well (to get a finite $\mathscr{I}$), Ω must approach a linear function, say $\Omega \sim a(a\xi+b)$. Using this in (60) and setting $\omega = a\xi+b$, $\hat{\theta} = a\theta$ one gets $d^2\hat{\theta}/d\omega^2-\omega^2\hat{\theta} = -\omega$, and the solutions of this which $\to 0$ at ∞ are of the form $\hat{\theta} = \theta_p(\omega)+c\theta_0(\omega)$, where

$$\theta_p(\omega) = \tfrac{1}{2}\omega \int_0^1 e^{-\frac{1}{2}\omega^2 t}(1-t^2)^{-\frac{1}{4}}\,dt \quad \text{and} \quad \theta_0(\omega) = \omega^{\frac{1}{2}} K_{\frac{1}{4}}(\tfrac{1}{2}\omega^2).$$

Of the six matching conditions, one is satisfied because of the first integral (62), and another is also satisfied automatically because it was possible in the numerical integration to determine quite accurately the relation between α and β for those solutions with finite $\mathscr{I}$. The remaining four matching conditions were used to determine a, b, c and α. The results of this computation are given in table 1, and the functions $\Theta(\xi)$ and $I_1 = -\int_0^\xi (1-\Omega\Theta)\,d\xi$, which correspond to the temperature deviations and the mean temperature field, are shown graphically in figure 2. The minimum value σ of $\mathscr{I}$ is 0·337, and for values of ξ larger than those given in table 1 a good approximation is:

$$\Theta \simeq 1{\cdot}431(\xi-0{\cdot}620)^{-1}, \quad 1-\Omega\Theta = -4{\cdot}095(\xi-0{\cdot}620)^{-4}.$$

Note that for large ξ $1-\Omega\Theta$ is negative (which is also obvious from (60) since Θ'' must be positive for large ξ), and that consequently the mean temperature gradient reverses its sign; however, where it is positive, it is numerically much smaller than its absolute value at the boundary. This result is perhaps not too surprising; away from the boundary the heat is transported almost entirely by convection, and in order to maximize heat transport one can perhaps accept a small adverse mean temperature gradient there, since this permits a somewhat

ξ	Θ	$1-\Omega\Theta$	Ω	ξ	Θ	$1-\Omega\Theta$	Ω	ξ	Θ
0	0	1	0	2·1	0·678	0·295	1·039	4·208	0·408
0·1	0·049	1·000	0·004	2·2	0·676	0·251	1·108	4·328	0·394
0·2	0·098	0·998	0·017	2·3	0·672	3·210	1·177	4·448	0·381
0·3	0·148	0·995	0·036	2·4	0·664	0·172	1·247	4·567	0·369
0·4	0·196	0·988	0·063	2·5	0·655	0·138	0·316	4·687	0·358
0·5	0·244	0·977	0·095	2·6	0·644	0·108	1·385	4·806	0·347
0·6	0·292	0·961	0·133	2·7	0·631	0·082	1·455	4·926	0·337
0·7	0·338	0·941	0·176	2·8	0·617	0·060	1·524	5·046	0·327
0·8	0·382	0·915	0.223	2·9	0·602	0·040	1·594	5·165	0·318
0·9	0·424	0·884	0·274	3·0	0·587	0·024	1·664	5·285	0·310
1·0	0·464	0·847	0·329	3·1	0·571	0·011	1·734	5·404	0·302
1·1	0·501	0·807	0·386	3·2	0·554	0·000	1·804	5·524	0·294
1·2	0·536	0·762	0·445	3·3	0·538	−0·008	1·875	5·644	0·287
1·3	0·566	0·713	0·507	3·4	0·522	−0·014	1·945	5·763	0·280
1·4	0·594	0·662	0·570	3·5	0·506	−0·019	2·014	5·883	0·274
1·5	0·617	0·608	0·635	3·6	0·491	−0·022	2·083	6·002	0·267
1·6	0·636	0·554	0·701	3·7	0·476	−0·024	2·152	6·122	0·262
1·7	0·652	0·500	0·767	3·8	0·461	−0·025	2·220	6·242	0·256
1·8	0·664	0·446	0·835	3·9	0·447	−0·026	2·291	6·361	0·250
1·9	0·672	0·393	0·902	4·0	0·434	−0·026	2·360	6·481	0·245
2·0	0·677	0·343	0·971	4·1	0·422	−0·025	2·430	6·601	0·240

TABLE 1

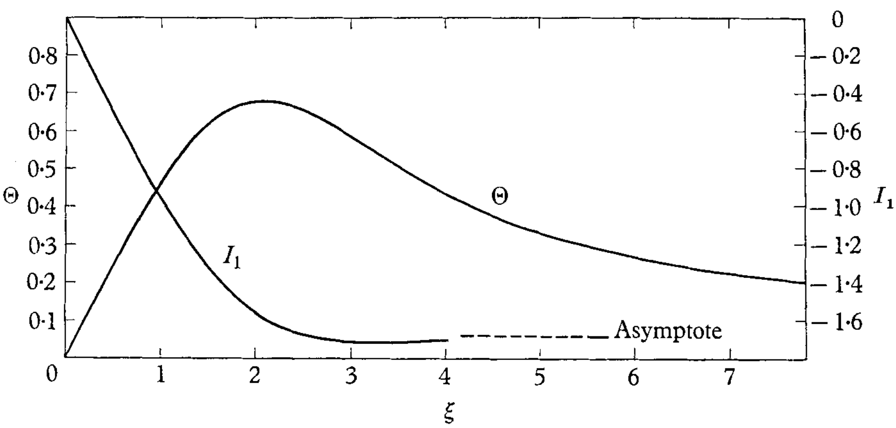

FIGURE 2. Temperature deviation (Θ) and mean temperature (I_1) functions for the maximizing flow of §4.

larger favourable gradient near the boundary where conduction is the primary mechanism of heat transport. One further point about this boundary-layer solution that should be mentioned is that the value of $\int_0^\infty (1-\Omega\Theta)\,d\xi$ is 5σ, as can easily be shown from the differential equations. This means (cf. (70) and

(71)) that the total change in mean temperature across the two boundary layers given by the above formulas is

$$2.\frac{\sigma^{-\frac{3}{2}}}{11}\left(\frac{R}{33}\right)^{\frac{3}{8}}.\sigma^{\frac{1}{2}}\left(\frac{R}{33}\right)^{-\frac{3}{8}}.5\sigma\Delta T = \tfrac{10}{11}\Delta T.$$

The reason for the discrepancy between this and the correct value ΔT is that we have used only the boundary-layer solution. The 'external flow' part of the solution (and the higher-order boundary-layer corrections), which we have not computed, has a negligible effect within the boundary layers, but over the whole layer does give a small but finite contribution to the mean temperature difference.

To what extent does the real flow resemble this maximizing 'flow'? A detailed resemblance is of course not to be expected; for one thing because the real convection is time dependent, whereas time disappears completely from the maximum problem, as soon as attention is restricted to the power integrals. (Incidentally the Prandtl number also is not present in the power integrals, so upper bounds obtained from them are independent of Prandtl number; in any case the observed dependence on Prandtl number is very weak—cf. Globe & Dropkin 1959.) On the other hand it is interesting to compare the mean properties of the real flow with the present results, and to a certain extent this can be done thanks to Townsend's (1959) experiments. Besides the ambiguities inherent in the non-physical character of the maximizing flow there are, however, also difficulties in interpreting Townsend's experiments in the present context, because the experiments were designed with the idea of studying convection into a semi-infinite region, and there was no top on the apparatus (other than the laboratory ceiling). Thus it is not clear what value should be taken for d in order to compare with the maximizing flow. Presumably all values of d between about 40 cm (near the top of the open box containing the convection system) and 4 or 5 m (roughly the height of the room) are equally plausible; however, while some finite d is required for comparison with the maximizing flow, the dependence on d is very weak, reflecting the fact that $\frac{3}{8}$ is nearly $\frac{1}{3}$. There is a similar ambiguity about ΔT; Townsend gives the difference in temperature between the heated lower plate and a point 40 cm above it, and this is in fact almost the same in all Townsend's experiments as the difference between the plate temperature and the mean temperature at any point more than 3 or 4 cm up. Thus this difference might be taken as ΔT; on the other hand, if the experimental observations are interpreted as corresponding to the lower boundary layer in a convection between two parallel plates, it would be more reasonable to take this difference to be $\frac{1}{2}\Delta T$. Suppose a rigid conducting plate held at the measured mean temperature at 40 cm were introduced at 40 cm. Would this greatly alter the measurements? It would certainly change the mean temperature profile, because the value at say 20 cm should become the average of the temperatures at the two plates, whereas before it was almost the same as at 40 cm, but whether or not the temperature fluctuations in the lower boundary layer would be similarly changed is not clear (though it seems likely). Despite these ambiguities of interpretation, whose resolution must await experiments with the same sort of detailed measurements as Townsend's but with two rigid plates, some interesting comparisons can be made. Perhaps the simplest is the magnitude of the temperature fluctuations. Both Townsend's observations of $(\overline{\theta^2})^{\frac{1}{2}}$, and the function T of the maximizing flow (which are the appropriate quantities to compare) rise to a maximum and then decrease slowly. We shall consider the detailed comparison of these curves presently, but the maximum amplitude of the temperature fluctuations is a quantity which we can compare without having to make a decision on the vertical scale. With each of his three standard heat fluxes Townsend found this maximum amplitude to be about $0{\cdot}6T\theta_0$ (his notation), and in each case this is nearly 18 % of $T_1 - T_a$, the difference of the plate temperature T_1 and the 'refer-

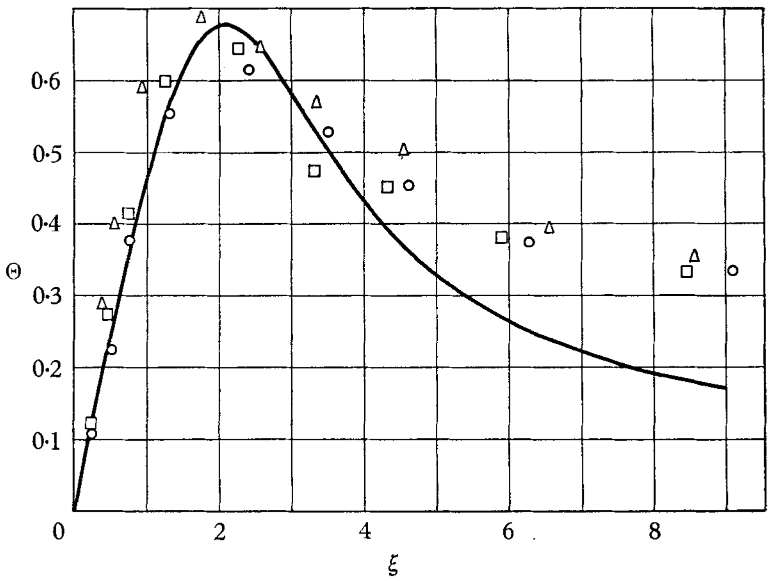

FIGURE 3. Comparison of temperature deviation with Townsend's experiments. See §5. ○, A; □, B; △, C.

ence temperature' T_a at 40 cm above the plate. The maximum of the function Θ is 0·678, and (cf. (69)) the maximum temperature fluctuation for the maximizing flow is $\Delta T/11\sigma$ times this, or $0{\cdot}183\Delta T$. Thus if it is appropriate to take $\Delta T = T_1 - T_a$, the real flow has the same maximum amplitude of temperature fluctuations as the maximizing flow. With $T_1 - T_a = \frac{1}{2}\Delta T$, which seems the more appropriate interpretation, the real temperature fluctuations are about *half* of those for maximum heat transport. Townsend did not measure velocity fluctuations, but if we guess that the observed vertical velocity fluctuations would also be about half of those for the maximizing flow, we should expect observed heat fluxes about $\frac{1}{4}$ of the maximum. With any reasonable estimate of d for Townsend's experiments, of the order of a metre, say, the Rayleigh number is about 10^8 or 10^9, and in this range it is in fact true that observed heat fluxes are about $\frac{1}{4}$ of the maximum given here. The observed profiles of mean temperature and of root-mean-square temperature fluctuations have, except for the previously mentioned factor of 2, a considerable resemblance to the maximizing profiles. To illustrate this I have attempted to compare Townsend's measurements with Θ and I_1. This cannot be done in a completely consistent manner, because of the factors of 2 and the fact that it is difficult to decide on an appropriate value of d. The comparison has been made as follows: ΔT was taken to be twice the difference between the temperature at the heated plate and the reference temperature 40 cm above it. Townsend gives the temperatures as functions of

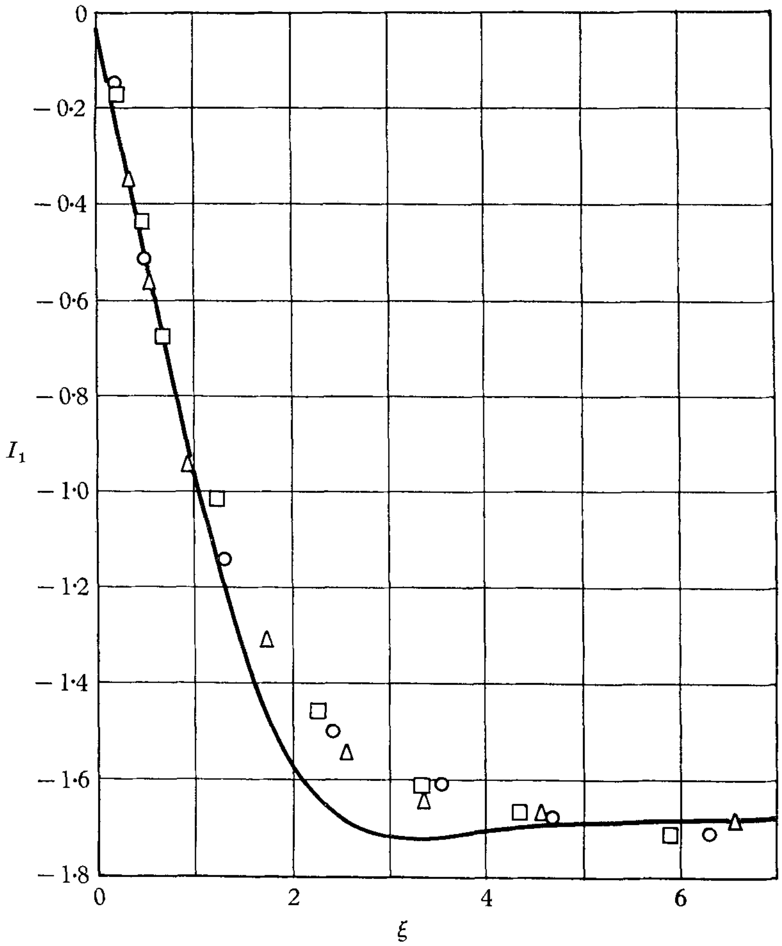

FIGURE 4. Comparison of mean temperature with Townsend's experiments. See §5. ○, A; □, B; △, C.

a dimensionless variable z/z_0 and this must be related to ξ; lacking a reliable value of d, this was done by matching the slopes of the mean temperature profiles at the plate. I was able to do this reasonably accurately because during my visit to Cambridge Dr Townsend very kindly loaned me his original notebook, and from the numerical data the slopes could be determined rather better than from the graphs given in the 1959 paper. Townsend's R.M.S. temperature fluctuations, $(\overline{\theta^2})^{\frac{1}{2}}$, should presumably correspond to $(11\sigma)^{-1}\,\Delta T\Theta$, or Θ should be $3{\cdot}71(\overline{\theta^2})^{\frac{1}{2}}/\Delta T$; as already remarked, this is experimentally about $\frac{1}{2}$ of the value computed for the maximizing flow and to facilitate comparison of the general shape of the profiles the values of Θ deduced from the measured $(\overline{\theta^2})^{\frac{1}{2}}$ have been arbitrarily doubled in plotting. It should be noted that a similar adjustment is implied by the method of determining ξ, since the observed heat flux is about $\frac{1}{4}$ of the maximum heat flux. Figure 3 shows Θ vs ξ with points deduced in this way from $(\overline{\theta^2})^{\frac{1}{2}}$ and z/z_0 for each of Townsend's three standard heat fluxes A, B, C. Figure 4 shows I_1 vs ξ, with similar points deduced from the measurements of mean temperature. The general shapes of the experimental and maximizing profiles are

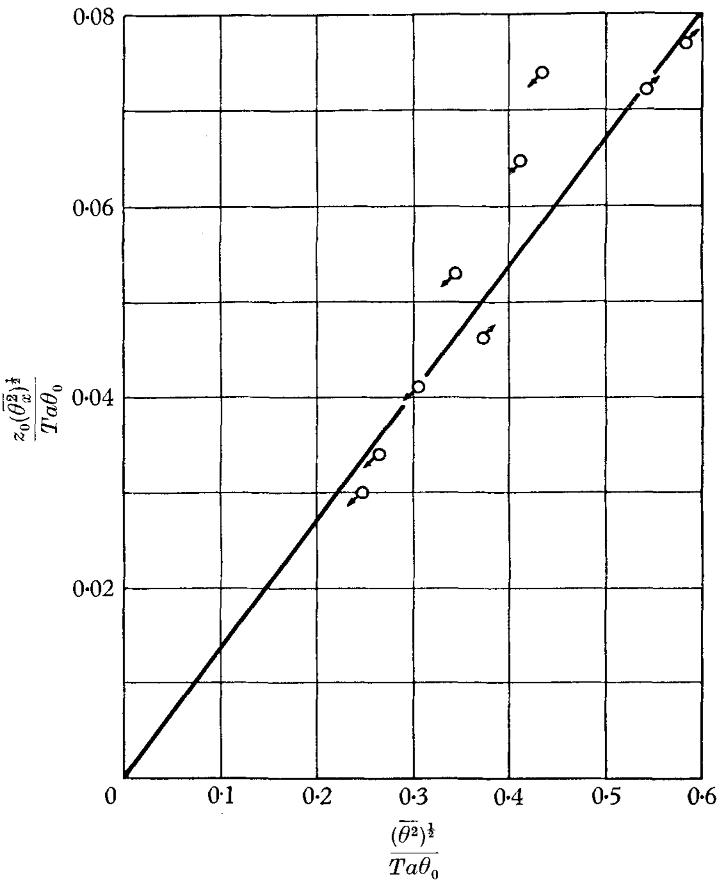

FIGURE 5. Horizontal scale for Townsend's heat flux B. Arrows indicate the direction of increasing z/z_0.

quite similar, the most notable difference other than the factor of 2 being that the actual R.M.S. temperature fluctuations fall off rather more slowly with distance from the boundary than does Θ. Townsend in fact found $(\overline{\theta^2})^{\frac{1}{2}} \propto (z/z_0)^{-0\cdot6}$ as the best power-law representation, whereas Θ falls off as ξ^{-1}.

One other feature of the maximizing flow that can be at least roughly compared with experiment is the horizontal scale. Townsend measured $(\overline{\theta_x^2})^{\frac{1}{2}}$ as well as $(\overline{\theta^2})^{\frac{1}{2}}$. If there were only one horizontal wave-number, plotting $(\overline{\theta_x^2})^{\frac{1}{2}}$ against $(\overline{\theta^2})^{\frac{1}{2}}$ should give a straight line. While this is not exactly so, there is a reasonably well-defined horizontal scale, as figure 5 shows in the case of Townsend's heat flux B, and the plots in cases A and C are similar. Estimating the slope on these plots and taking $\partial/\partial x = 2^{-\frac{1}{2}}a$, all three cases give $a \simeq 1{\cdot}8\,\mathrm{cm}^{-1}$. If we knew d, this could be compared with the maximizing wave-number $(R/33)^{\frac{1}{4}}\,d^{-1}$, but in view of the uncertainty with regard to d, it seems better to compute d by assuming the observed wave-number $1{\cdot}8\,\mathrm{cm}^{-1}$ is equal to

$$(R/33)^{\frac{1}{4}}\,d^{-1} \simeq (3{\cdot}1\Delta T d^{-1})^{\frac{1}{4}} \text{ (c.g.s. units)};$$

this gives values of d of the order of 30 cm, which seems to be reasonable. d cannot be obtained very precisely by such a matching because it depends on the fourth power of the rather imprecise 'observed' value of a, but in any case the actual horizontal scale seems to be of the same order of magnitude as that required for maximum heat transport. Another possible comparison would be to use the formulas of the maximizing flow to determine d from the observed values of ΔT and heat flux. Since $\frac{3}{8}$ is quite close to $\frac{1}{3}$, however, d obtained this way is extremely sensitive to uncertainties in ΔT and the flux (d is proportional to $(\mathrm{flux})^{8}\,(\Delta T)^{11}$) and about all that can be said is that if reasonable values for ΔT and d are selected, one obtains Rayleigh numbers of the order of 10^9 and heat fluxes about four times Townsend's measurements.

These comparisons, particularly the detailed comparison of the profiles, should not of course be taken too seriously, partly because of the factor 2, but especially because in principle the maximizing 'flow' is physically artificial, and in many details does not, and cannot, resemble the real flow at all—particularly with respect to time dependence. Nevertheless, the fact that the mean properties of the real motion do exhibit a considerable similarity with those of the artificial flow which maximizes heat flux subject only to the overall balance of energy and entropy, and continuity, seems to me to lend some support to the hypothesis that the actual motion which occurs is, or anyway approximates, a solution of the Boussinesq equations which maximizes heat transport.

The first part of this work was done during the summer of 1961 while I was participating in the summer programme in Geophysical Fluid Dynamics at the Woods Hole Oceanographic Institution, supported by the National Science Foundation. It was continued and substantially completed during my visit to Cambridge during the following academic year, a visit made possible by the generous hospitality of Dr Batchelor and the other Cambridge applied mathematicians, as well as a grant from the Guggenheim foundation. Dr Swinnerton-Dyer kindly arranged for the preliminary computations of Θ and Ω to be done at the Computation Laboratory at Cambridge. The final computations were done at the Computation Laboratory of Stockholm University with the assistance of P. Crutzen; I am particularly indebted to Dr Pierre Welander for his hospitality during my visit to Stockholm in the summer of 1962. This work has also been partially supported by the Office of Naval Research.

REFERENCES

Chandrasekhar, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Oxford: Clarendon Press.

Globe, S. & Dropkin, D. 1959 *Trans. Amer. Soc. Mech. Engrs, J. Heat Transf.* **81**, 24.

Gor'kov, L. P. 1958 *Sov. Phys. JETP*, **6**, 311.

Kraichnan, R. 1962 *Phys. Fluids*, **5**, 1374.

Malkus, W. V. R. 1954*a* *Proc. Roy. Soc.* A, **225**, 185.

Malkus, W. V. R. 1954*b* *Proc. Roy. Soc.* A, **225**, 196.

Malkus, W. V. R. & Veronis, G. 1958 *J. Fluid Mech.* **4**, 225.

Mihaljan, J. 1960 *Tech. Rep. no.* 1 *on Thermal Convection in Rotating Systems*. Nat. Sci. Foundation.

Pellew, A. & Southwell, R. V. 1940 *Proc. Roy. Soc.* A, **176**, 312.

Spiegel, E. & Veronis, G. 1960 *Astrophys. J.* **131**, 442.

Townsend, A. A. 1959 *J. Fluid Mech.* **5**, 209.

Townsend, A. A. 1962 *Mécanique de la turbulence*, p. 167. Centre National de la Recherche Scientifique, Paris.